Malaysian Journal of Mathematical Sciences 18(3): 553–566 (2024)
https://doi.org/10.47836/mjms.18.3.06

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Strong Solutions of Brusselator System

Khitam, A. H. A.[1] and Al-Juaifri, G. A.*[2]

[1]*Department of Mathematics, Faculty of Computer Science and Mathematics, University of Kufa, Kufa, Iraq*

E-mail: ghassana.ali@uokufa.edu.iq
**Corresponding author*

Received: 16 March 2024
Accepted: 20 May 2024

Abstract

The study involves a mathematical analysis of the Brusselator system on a convex bounded three-dimensional open domain, considering Neumann boundary conditions. We establish the global existence and uniqueness of the strong solution for this system. Achieving high regularity for the strong solution requires stringent conditions on the initial data. The study demonstrates the continuous dependence of the solution on the initial conditions.

Keywords: Brusselator system; existence; Faedo-Galerkin; Neumann boundary conditions; strong solution.

1 Introduction

In practical scenarios, addressing the chemical reactions within systems involving two variable intermediates and various initial and final products, whose concentrations are regulated throughout the reaction mechanism, is a crucial challenge. This observation is articulated by Nicolis and Prigogine in their work [2, 11, 14]. The rate equations necessitate the inclusion of at least a cubic nonlinearity [12]. The trimolecular model, also known as the Brusselator [18], proves to be a valuable tool for exploring chemical kinetics processes. It is employed to study ozone formation by a three-step collision involving a tri-molecular reaction with atomic oxygen. This model finds applications in enzymatic reactions, as well as in establishing connections between specific modes in plasma physics and lasers. Here, we investigate the Brusselator system with Neumann boundary conditions in the form:(see [17, 4])

$$\frac{\partial \vartheta}{\partial t} - \mu_1 \Delta \vartheta - \mu_2 \Delta \varphi = -(\alpha+1)\vartheta + \vartheta^2 \varphi, \quad \text{in} \quad \Omega_T, \tag{1}$$

$$\frac{\partial \varphi}{\partial t} - \mu_3 \Delta \varphi = \alpha \vartheta - \vartheta^2 \varphi, \quad \text{in} \quad \Omega_T, \tag{2}$$

$$\frac{\partial \vartheta}{\partial \nu} = 0, \quad \frac{\partial \varphi}{\partial \nu} = 0, \quad \text{on} \quad S_T, \tag{3}$$

$$\vartheta(\cdot, 0) = \vartheta_0, \quad \varphi(\cdot, 0) = \varphi_0, \quad \text{in} \quad \Omega, \tag{4}$$

where $\Omega_T = \Omega \times (0, T)$, (Ω) represents a bounded domain in $\mathbb{R}^L (L = 1, 2, 3)$; where smooth boundary $\partial\Omega$, $S_T = \partial\Omega \times (0, T)$, ν represents the exterior unit normal to $\partial\Omega$, ϑ_0 and φ_0 denote the initial data, and $\alpha > 0$ is a positive parameter. An essential feature of the model is that the positive diffusion constants μ_1, μ_2, μ_3 typically satisfy $\mu_1\mu_3 > \mu_2^2$, and in some cases with $\mu_2 \ll 1$. On the boundary, α is dimensionless non-negative constant. The above coupled nonlinear system arises in chemical kinetics, as well as in thermodynamics and in pattern formation. There are a lot of recent studies that contain the study of this system in theory or numerically, which include a lot of important applications associated with this system [9, 10].

The significance of investigating interaction propagation systems with Neumann boundary conditions is underscored by various factors. While extensive research has been conducted on interaction propagation systems with both Neumann and Dirichlet conditions of has been relatively limited. Sherratt [15] introduces the concept of "oscillatory" reaction-diffusion equations with applications ecology. Indeed, numerous recent studies have delved into the investigation of interaction propagation systems with Neumann boundary conditions [3, 8].

2 Notation and Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^L$, $L \leq 3$ with boundary $\partial\Omega$. For $L = 1, 2, 3$, we assume that $\partial\Omega$ is a Lipschitz boundary. In this paper, we utilize the standard notation for Sobolev spaces, representing the norm of $W^{\lambda,\gamma}(\Omega)$, where $\lambda \in \mathbb{N}$ and $\gamma \in [1, \infty]$ by $\| \cdot \|_{\lambda,\gamma}$ and the semi-norm by $| \cdot |_{\lambda,\gamma}$. When $\gamma = 2$, we denote $W^{\lambda,2}(\Omega)$ as $H^\lambda(\Omega)$ with the norm $\| \cdot \|_\lambda$ and semi-norm $| \cdot |_\lambda$. For $\lambda = 0$, $W^{0,2}(\Omega)$ is equivalent to $L^2(\Omega)$. The $L^2(\Omega)$ inner product over Ω with norm $\| \cdot \|_0 = | \cdot |_0$ is represented by $(\cdot,\cdot)$. Additionally, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$, where $(H^1(\Omega))'$ is the dual space of $H^1(\Omega)$. The norm on $(H^1(\Omega))'$ is defined by

$$\|\psi\|_{(H^1(\Omega))'} := \sup_{\kappa\neq 0}\frac{|\langle\psi,\kappa\rangle|}{\|\kappa\|_1} \equiv \sup_{\|\kappa\|_1=1}\|\langle\psi,\kappa\rangle\|. \tag{5}$$

Consider Y as a Banach space, where $(1\leq\gamma\leq\infty)$. Let $L^\gamma(0,T;Y)$ represents the Banach space of all measurable functions $\psi(t):[0,T]\to Y$ such that $t\to\|\psi(t)\|_Y$ is in $L^\gamma(0,T)$ with the norm

$$\|\psi(t)\|_{L^\gamma(0,T;Y)} = \left(\int_0^T |\psi(t)|_Y^\gamma dt\right)^{\frac{1}{\gamma}},$$

$$\|\psi(t)\|_{L^\infty(0,T;Y)} = ess\sup_{t\in(0,T)}\|\psi(t)\|_Y.$$

We also define $L^\gamma(\Omega_T) = L^\gamma(0,T;L^\gamma(\Omega))$. Additionally, we define $C([0,T];Y)$ as the space of continuous functions from $[0,T]$ into Y, consisting of $\psi(t):[0,T]\to Y$ such that $\psi(t)\to\psi(t_0)$ in Y as $t\to t_0$. It is important to note that $C([0,T];Y)$ is a Banach space with the associated norm [16]:

$$\|\psi(t)\|_{C([0,T];Y)} = \sup_{t\in[0,T]}\|\psi(t)\|_Y,$$

and recall well-known Sobolev results:

$$H^1(\Omega)\overset{c}{\hookrightarrow} L^\gamma(\Omega)\hookrightarrow (H^1(\Omega))' \text{ holds for } \gamma\in\begin{cases}[1,\infty] & \text{if} \quad L=1,\\ [1,\infty) & \text{if} \quad L=2,\\ [1,6] & \text{if} \quad L=3,\end{cases} \tag{6}$$

where '$\hookrightarrow$' denotes the continuous embedding. The embedding in (6) is compact, as per the Rellich-Kondrachov theorem (refer to, for example, [5] page 8), with $\gamma\in[1,6]$ replaced by $\gamma\in[1,6)$ in this case when $L=3$. This compact embedding is denoted by the symbol $\overset{c}{\hookrightarrow}$.

The Hölder's inequality is also required frequently: For $1\leq c,d\leq\infty$ such that $\frac{1}{c}+\frac{1}{d}=1$; if $\phi\in L^c(\Omega)$ and $\psi\in L^d(\Omega)$, then $\psi\phi\in L^1(\Omega)$ and

$$\|\psi\phi\|_{0,1}\leq\|\psi\|_{0,c}\|\phi\|_{0,d}. \tag{7}$$

Applying the Hölder's inequality twice gives

$$\|\psi\phi\theta\|_{0,1}\leq\|\psi\|_{0,c}\|\phi\|_{0,d}\|\theta\|_{0,e}, \quad \text{for} \quad 1\leq c,d,e\leq\infty \quad \text{such that} \quad \frac{1}{c}+\frac{1}{d}+\frac{1}{e}=1. \tag{8}$$

For other purposes, we remember the Sobolev interpolation theorem [1]: Let $\psi\in W^{\lambda,\gamma}(\Omega)$, for $\gamma\in[1,\infty]$, $\lambda\geq 1$, then there are constants A and $\epsilon=\frac{L}{\lambda}\left(\frac{1}{q}-\frac{1}{s}\right)$ such that the following Gagliardo-Nirenberg inequality holds by [16]:

$$\|w\|_{0,s}\leq A\|w\|_{0,q}^{1-\epsilon}\|w\|_{\lambda,q}^{\epsilon} \quad \text{holds for} \quad s\in\begin{cases}[q,\infty] & \text{if} \quad \lambda-\frac{L}{q}>0,\\ [q,\infty) & \text{if} \quad \lambda-\frac{L}{q}=0,\\ [q,-\frac{L}{\lambda-L/q}] & \text{if} \quad \lambda-\frac{L}{q}<0.\end{cases} \tag{9}$$

We also require the following Grönwall lemma in differential form: Let $\Gamma(t)\in W^{1,1}(0,T)$ and $\varphi_1(t),\varphi_2(t),\varphi_3(t)\in L^1(0,T)$, where all functions are non-negative. It follows from

$$\frac{d\Gamma(t)}{dt}+\varphi_2(t)\leq\varphi_3(t)\Gamma(t)+\varphi_1(t) \quad \text{a.e.} \quad t\in[0,T],$$

that

$$\Gamma(T)+\int_0^T \varphi_2(t)dt \leq exp(\int_0^T \varphi_3(\tau)d\tau)\Gamma(0)+exp(\int_0^T \varphi_3(\tau)d\tau)\int_0^T \varphi_1(\tau)d\tau. \tag{10}$$

We will be frequently need Young's inequality in the form

$$\nu_1\nu_2 \leq \varepsilon^{\frac{\varepsilon_1}{\varepsilon_2}}\frac{\nu_1^{\varepsilon_1}}{\varepsilon_1}+\varepsilon^{-1}\frac{\nu_2^{\varepsilon_2}}{\varepsilon_2}, \quad \frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}=1, \tag{11}$$

that valids for any $\nu_1, \nu_2 \geq 0$, $\varepsilon > 0$ and $\varepsilon_1, \varepsilon_2 > 1$. Another valuable implication of Young's inequality is as follows:

$$\nu_1\nu_2 \geqslant -\varepsilon\frac{\nu_1^2}{2}-\varepsilon^{-1}\frac{\nu_2^2}{2}, \quad \forall \nu_1, \nu_2 \in \mathbb{R}, \forall \varepsilon > 0. \tag{12}$$

3 Strong Solutions

Let $\{z_i\}_{i=1}^{\infty}$ be orthogonal basis for $H^1(\Omega)$ and orthonormal basis for $L^2(\Omega)$, depending on the Neumann problem:

$$-\Delta z_i + z_i = \mu_i z_i, \quad \text{in} \quad \Omega, \qquad \frac{\partial z_i}{\partial \nu}=0, \quad \text{on} \quad \partial\Omega, \tag{13}$$

where

$$1 \leq \mu_1 \leq \mu_2 \leq \mu_3 \leq \dots \leq \mu_k \leq \dots \quad \text{with} \quad \lim_{i\to\infty}\mu_i=\infty, \tag{14}$$

is an infinite set of corresponding eigenvalues. Normalising so that $(z_i, z_j)_{H^1(\Omega)} = \mu_i\delta_{ij}$ and $(z_i, z_j)_{L^2(\Omega)} = \delta_{ij}$. Let V^k denote the finite dimensional subspace of $H^1(\Omega)$ spanned by $\{z_i\}_{i=0}^k$. We define the L^2 projection onto V^k, $P^k : L^2(\Omega) \longmapsto V^k$, by $P^k\upsilon = \sum_{j=1}^k(\upsilon, z_j)z_j$, we also notice that $(P^k\upsilon, \eta^k) = (\upsilon, \eta^k)$ for all $\eta^k \in V^k$. This definition clearly makes sense for elements of $H^1(\Omega) \subset L^2(\Omega)$:

$$(\frac{\partial\vartheta^k}{\partial t},\eta^k)+\mu_1(\nabla\vartheta^k,\nabla\eta^k)+\mu_2(\nabla\varphi^k,\nabla\eta^k) = -(\alpha+1)(\vartheta^k,\eta^k)+((\vartheta^k)^2\varphi^k,\eta^k) \quad \forall\eta^k \in V^k, \tag{15}$$

$$(\frac{\partial\varphi^k}{\partial t},\eta^k)+\mu_3(\nabla\varphi^k,\nabla\eta^k) = \alpha(\vartheta^k,\eta^k)-((\vartheta^k)^2\varphi^k,\eta^k) \quad \forall\eta^k \in V^k. \tag{16}$$

Theorem 3.1. *Assume $\Omega \subset \mathbb{R}^L$ is bounded, convex , and open domain with a boundary $\partial\Omega$ of class C^2, suppose that ϑ_0, $\varphi_0 \in H^1(\Omega)$ and*

$$\begin{aligned}
&\vartheta(x,t) \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap L^4(\Omega_T),\\
&\varphi(x,t) \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),\\
&\vartheta(x,t)+\varphi(x,t) \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),
\end{aligned} \tag{17}$$

then the system $(\mathfrak{G})$ *have a unique strong solution* $\{\vartheta, \varphi\}$ *satisfying*

$$\vartheta(x,t) \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^4(\Omega)), \\ \cap L^\infty(0,T;L^6(\Omega)) \cap L^6(\Omega_T) \cap L^8(\Omega_T) \cap C([0,T];H^1(\Omega)),$$

$$\varphi(x,t) \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^4(\Omega)), \\ \cap L^\infty(0,T;L^6(\Omega)) \cap L^4(\Omega_T) \cap L^6(\Omega_T) \cap C([0,T];H^1(\Omega)),$$

$$\frac{\partial \vartheta}{\partial t}, \frac{\partial \varphi}{\partial t} \in L^2(\Omega_T),$$

and the system $(\mathfrak{G})$ *hold as equalities in* $L^2(\Omega_T)$*. Furthermore,*

$$(\vartheta_0(.), \varphi_0(.)), \longmapsto (\vartheta(., t; \vartheta_0, \varphi_0), \varphi(., t; \vartheta_0, \varphi_0)),$$

is continuous in $H^1(\Omega)$*.*

Proof. Proving Theorem 3.1 will take place in several stages. □

3.1 Existence

Estimate I . Setting $\eta^k = -\Delta\vartheta^k$, $\eta^k = -\Delta\varphi^k$ in the (15) and (16), integrating by parts, it follows that

$$\frac{1}{2}\frac{d}{dt}\|\nabla\vartheta^k\|_0^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\varphi^k\|_0^2 + \mu_1\|\Delta\vartheta^k\|_0^2 + \mu_3\|\Delta\varphi^k\|_0^2 \\ = -\mu_2(\Delta\varphi^k, \Delta\vartheta^k) - \alpha(\vartheta^k, \Delta\varphi^k) + (\alpha+1)(\vartheta^k, \Delta\vartheta^k) - (\varphi^k(\vartheta^k)^2, \Delta\vartheta^k) + (\varphi^k(\vartheta^k)^2, \Delta\varphi^k). \quad (18)$$

By integrating by parts and use of (11), we obtain that

$$\begin{aligned} -(\varphi^k(\vartheta^k)^2, \Delta\vartheta^k) =& (\nabla(\varphi^k(\vartheta^k)^2, \nabla\vartheta^k) \\ =& \int_\Omega \varphi^k(2\vartheta^k\nabla\vartheta^k)\nabla\vartheta^k + (\vartheta^k)^2\nabla\varphi^k\nabla\vartheta^k dx \\ \le& 2\|\varphi^k\|_{0,\infty}\|\vartheta^k\|_{0,\infty}\int_\Omega |\nabla\vartheta^k|^2 dx + \|\vartheta^k\|_{0,\infty}^2\int_\Omega \nabla\vartheta^k\nabla\varphi^k dx \\ \le& \left[\|\vartheta^k\|_{0,\infty}^2 + \|\varphi^k\|_{0,\infty}^2\right]\left[|\vartheta^k|_1^2\right] + \frac{1}{2}\left[\|\vartheta^k\|_{0,\infty}^2\right]\left[|\vartheta^k|_1^2 + |\varphi^k|_1^2\right] \\ \le& C\left[\|\vartheta^k\|_{0,\infty}^2 + \|\varphi^k\|_{0,\infty}^2\right]\left[|\vartheta^k|_1^2 + |\varphi^k|_1^2\right]. \end{aligned} \quad (19)$$

By integrating by parts, and using (11) and (17), we have that

$$\begin{aligned} (\varphi^k(\vartheta^k)^2, \Delta\varphi^k) =& -(\nabla(\varphi^k(\vartheta^k)^2), \nabla\varphi^k) \\ =& -\int_\Omega (\vartheta^k)^2\nabla\varphi^k\nabla\varphi^k + \varphi^k(2\vartheta^k)\nabla\vartheta^k\nabla\varphi^k dx \\ \le& -\int_\Omega |\vartheta^k\nabla\varphi^k|^2 dx + \frac{1}{2}\int_\Omega |\vartheta^k\nabla\varphi^k|^2 dx + 2\int_\Omega |\varphi^k\nabla\vartheta^k|^2 dx \\ =& \frac{-1}{2}\|\vartheta^k\nabla\varphi^k\|_0^2 + 2\|\varphi^k\|_{0,\infty}|\vartheta^k|_1^2. \end{aligned} \quad (20)$$

Now, by using (11) and (17) for the second and third terms in the right side of (18), we have that

$$(\alpha+1)(\vartheta^k,\Delta\vartheta^k)-\alpha(\vartheta^k,\Delta\varphi^k)\leq-(\alpha+1)\|\nabla\vartheta^k\|_0^2+\frac{\alpha^2}{2\mu_3}\|\vartheta^k\|_0^2+\frac{\mu_3}{2}\|\Delta\varphi^k\|_0^2. \tag{21}$$

Compensating (19)-(21) into (18), and multiply the result by two, we get to

$$\begin{aligned}\frac{d}{dt}\|\nabla\vartheta^k\|_0^2+\frac{d}{dt}\|\nabla\varphi^k\|_0^2+\mu_1\|\Delta\vartheta^k\|_0^2+(2\mu_3-\frac{\mu_2^2}{\mu_1})\|\Delta\varphi^k\|_0^2+2(\alpha+1)\|\nabla\vartheta^k\|_0^2+\|\vartheta^k\nabla\varphi^k\|_0^2\\ \leq\frac{\alpha^2}{\mu_3}\|\vartheta^k\|_0^2+C\big[\|\vartheta^k\|_{0,\infty}^2+\|\varphi^k\|_{0,\infty}^2\big]\big[|\vartheta^k|_1^2+|\varphi^k|_1^2\big].\end{aligned} \tag{22}$$

Application of Grönwall lemma gives that

$$\begin{aligned}&\|\nabla\vartheta^k(T)\|_0^2+\|\nabla\varphi^k(T)\|_0^2+(2\mu_3-\frac{\mu_2^2}{\mu_1})\|\vartheta^k\|_{L^2(0,T;H^2(\Omega))}^2+\mu_3\|\varphi^k\|_{L^2(0,T;H^2(\Omega))}^2\\&+2(\alpha+1)\|\vartheta^k\|_{L^2(0,T;H^1(\Omega))}^2+\|\vartheta^k\nabla\varphi^k\|_{L^2(\Omega_T)}^2\\&\leq\frac{\alpha^2}{\mu_3}\|\vartheta^k\|_{L^2(\Omega_T)}^2+C\big[\|\vartheta^k\|_{L^2(0,T;L^\infty(\Omega))}^2+\|\varphi^k\|_{L^2(0,T;L^\infty(\Omega))}^2\big]\\&\big[\|\vartheta^k\|_{L^2(0,T;H^1(\Omega))}^2+\|\varphi^k\|_{L^2(0,T;H^1(\Omega))}^2\big]+\|\nabla\vartheta^k(0)\|_0^2+\|\nabla\varphi^k(0)\|_0^2.\end{aligned} \tag{23}$$

By the uniform bounds in (17), the injection $L^2(0,T;H^1(\Omega))\hookrightarrow L^2(0,T;L^\infty(\Omega))$, we have that (23) is bounded. Thus, we have that ϑ^k,φ^k are uniformly bounded in $L^\infty(0,T;H^1(\Omega))$. From the fact that $L^1(0,T;H^1(\Omega)^{'})$ is the pre-dual of $L^\infty(0,T;H^1(\Omega))$, which is not reflexive Banach space, we conclude from the first two edges (23) that

$$\vartheta^k\rightharpoonup^*\vartheta,\quad\text{in}\quad L^\infty(0,T;H^1(\Omega)), \tag{24}$$

$$\varphi^k\rightharpoonup^*\varphi,\quad\text{in}\quad L^\infty(0,T;H^1(\Omega)). \tag{25}$$

Then, we have $\{\vartheta,\varphi\}\in L^\infty(0,T;H^1(\Omega))$. Some known elliptical regularity results are applied to limited, convex and open domains. From the eigenvalue in (13) and (see [7], Theorem 3.2.1.3), we have that for a fixed (finite) k, $z_i\in H^2(\Omega)$, for $i=1,\ldots,k$. Hence, $\vartheta^k(.,t),\varphi^k(.,t)\in H^2(\Omega)$ for almost every (a.e.) $t\in(0,T)$. Thus, by [7] we have $\|\vartheta^k\|_2\leq C\|\Delta\vartheta^k\|_0$, for some positive constant C and a.e. $t\in(0,T)$. Therefore, from the third and fourth bounds in (23), we conclude that ϑ^k,φ^k are uniformly bounded in $L^2(0,T;H^2(\Omega))$. Since $L^2(0,T;H^2(\Omega))$ is a reflexive Banach space [19], then, by compactness arguments [6], we get the existence of subsequences $\{\vartheta^k,\varphi^k\}\in L^2(0,T;H^2(\Omega))$ such that

$$\vartheta^k\rightharpoonup\vartheta,\quad\text{in}\quad L^2(0,T;H^2(\Omega)), \tag{26}$$

$$\varphi^k\rightharpoonup\varphi,\quad\text{in}\quad L^2(0,T;H^2(\Omega)). \tag{27}$$

Thus, we have $\{\vartheta,\varphi\}\in L^2(0,T;H^2(\Omega))$. Furthermore, since $\frac{\partial\vartheta^k}{\partial\nu}=0$ and $\frac{\partial\varphi^k}{\partial\nu}=0$ on $\partial\Omega$, and given the weak convergence of $\vartheta^k\to\vartheta$ and $\varphi^k\to\varphi$ in $H^2(\Omega)$, it follows that $\frac{\partial\vartheta}{\partial\nu}=0$ and $\frac{\partial\varphi}{\partial\nu}=0$ on $L^2(\partial\Omega)$.

Estimate II. Set $\eta^k=\frac{\partial\vartheta^k}{\partial t}$, $\eta^k=\frac{\partial\varphi^k}{\partial t}$ in the weak form (15)-(16), combine the results, we have that

$$\begin{aligned}\|\frac{\partial \vartheta^k}{\partial t}\|_0^2+\|\frac{\partial \varphi^k}{\partial t}\|_0^2+\frac{\mu_1}{2}\frac{d}{dt}\|\nabla\vartheta^k\|_0^2+\frac{\mu_3}{2}\frac{d}{dt}\|\nabla\varphi^k\|_0^2 =&\alpha\big(\vartheta^k,\frac{\partial \varphi^k}{\partial t}\big)-(\alpha+1)(\vartheta^k,\frac{\partial \vartheta^k}{\partial t})\\ &+\big(\varphi^k(\vartheta^k)^2,\frac{\partial \vartheta^k}{\partial t}\big)-\big(\varphi^k(\vartheta^k)^2,\frac{\partial \varphi^k}{\partial t}\big)\\ &-\mu_2(\nabla\varphi^k,\nabla\frac{\partial \vartheta^k}{\partial t}).\end{aligned}\tag{28}$$

By using (11), we get that

$$\alpha\big(\vartheta^k,\frac{\partial \varphi^k}{\partial t}\big)\le\alpha^2\|\vartheta^k\|_0^2+\frac{1}{4}\|\frac{\partial \varphi^k}{\partial t}\|_0^2.\tag{29}$$

By using (7) and (11), we obtain that

$$-\big(\varphi^k(\vartheta^k)^2,\frac{\partial \varphi^k}{\partial t}\big)\le\|\varphi^k\|_{0,\infty}\|\vartheta^k\|_{0,4}^2\|\frac{\partial \varphi^k}{\partial t}\|_0\le\|\varphi^k\|_{0,\infty}^2\|\vartheta^k\|_{0,4}^4+\frac{1}{4}\|\frac{\partial \varphi^k}{\partial t}\|_0^2,\tag{30}$$

$$\big(\varphi^k(\vartheta^k)^2,\frac{\partial \vartheta^k}{\partial t}\big)\le\|\varphi^k\|_{0,\infty}\|\vartheta^k\|_{0,4}^2\|\frac{\partial \vartheta^k}{\partial t}\|_0\le\|\varphi^k\|_{0,\infty}^2\|\vartheta^k\|_{0,4}^4+\frac{1}{4}\|\frac{\partial \vartheta^k}{\partial t}\|_0^2.\tag{31}$$

Combining (29)-(31) and multiplying by 2, implies that

$$\begin{aligned}&\|\frac{\partial \vartheta^k}{\partial t}\|_0^2+\|\frac{\partial \varphi^k}{\partial t}\|_0^2+\mu_1\frac{d}{dt}\|\nabla\vartheta^k\|_0^2+\mu_3\frac{d}{dt}\|\nabla\varphi^k\|_0^2+(\alpha+1)\frac{d}{dt}\|\vartheta^k\|_0^2\\ &\qquad\le 2\alpha^2\|\vartheta^k\|_0^2+\mu_2^2\|\Delta\varphi^k\|_0^2+4\big[\|\varphi^k\|_{0,\infty}^2\|\vartheta^k\|_{0,4}^4\big].\end{aligned}\tag{32}$$

Integrating over time, it follows that

$$\begin{aligned}&\int_0^T\|\frac{\partial \vartheta^k}{\partial t}\|_0^2dt+\int_0^T\|\frac{\partial \varphi^k}{\partial t}\|_0^2dt+\mu_1\big\|\nabla\vartheta^k(T)\big\|_0^2+\mu_3\big\|\nabla\varphi^k(T)\big\|_0^2+(\alpha+1)\|\vartheta^k(T)\|_0^2\\ &\le 2\alpha^2\int_0^T\|\vartheta^k\|_0^2dt+2\mu_2^2\int_0^T\|\Delta\varphi^k\|_0^2dt+4\int_0^T\big[\|\varphi^k\|_{0,\infty}^2\|\vartheta^k\|_{0,4}^4\big]dt+\mu_1\big\|\nabla\vartheta^k(0)\big\|_0^2\\ &+\mu_3\big\|\nabla\varphi^k(0)\big\|_0^2+(\alpha+1)\|\vartheta^k(0)\|_0^2.\end{aligned}\tag{33}$$

By using (11), and $L^\infty(0,T;H^1(\Omega))\hookrightarrow L^\infty(\Omega_T)$ on the right side of (33) leads to

$$\begin{aligned}\int_0^T\|\varphi^k\|_{0,\infty}^2\|\vartheta^k\|_{0,4}^4dt\le&\|\varphi^k\|_{L^\infty(\Omega_T)}^2\|\vartheta^k\|_{L^4(\Omega_T)}^4\\ \le&\frac{1}{2}\|\varphi^k\|_{L^\infty(0,T;H^1(\Omega))}^4+\frac{1}{2}\|\vartheta^k\|_{L^4(\Omega_T)}^8.\end{aligned}\tag{34}$$

By noting (34) and the bounds in (17), we see that (33) is bounded by a positive constant. We get that $\frac{\partial \vartheta^k}{\partial t}$ and $\frac{\partial \varphi^k}{\partial t}$ are uniformly bounded in $L^2(\Omega_T)$. As $L^2(\Omega_T)$ is a reflexive Banach space, thus, by compactness arguments, we get to the existence of subsequences $\{\vartheta^k,\varphi^k\}\in L^2(\Omega_T)$ such that

$$\frac{\partial \vartheta^k}{\partial t}\rightharpoonup\frac{\partial \vartheta}{\partial t},\quad\text{in}\quad L^2(\Omega_T),\tag{35}$$

$$\frac{\partial \varphi^k}{\partial t}\rightharpoonup\frac{\partial \varphi}{\partial t},\quad\text{in}\quad L^2(\Omega_T).\tag{36}$$

Thus, we have that $\frac{\partial \vartheta}{\partial t}, \frac{\partial \varphi}{\partial t} \in L^2(\Omega_T)$.

Estimate III. Setting $\eta^k = (\varphi^k)^3$ in (16), it follows that

$$\frac{1}{4}\frac{d}{dt}\|\varphi^k\|_{0,4}^4 + 3\mu_3\|\varphi^k\nabla\varphi^k\|_0^2 + \|\vartheta^k(\varphi^k)^2\|_0^2 = \alpha(\vartheta^k, (\varphi^k)^3). \tag{37}$$

By using (11) on (37), we have that

$$\alpha(\vartheta^k, (\varphi^k)^3) \leq \frac{3}{4}\|\vartheta^k(\varphi^k)^2\|_0^2 + \frac{\alpha^2}{3}\|\varphi^k\|_0^2. \tag{38}$$

By substituting (38) in (37), multiplying the result by 4, and integrating over time, we have that

$$\|\varphi^k(T)\|_{0,4}^4 + 12\mu_3\|\varphi^k\nabla\varphi^k\|_{L^2(\Omega_T)}^2 + \|\vartheta^k(\varphi^k)^2\|_{L^2(\Omega_T)}^2 \leq \frac{4\alpha^2}{3}\|\varphi^k\|_{L^2(\Omega_T)}^2 + \|\varphi^k(0)\|_{0,4}^4. \tag{39}$$

Recalling $\varphi_0^k \in H^1(\Omega)$, using (17) and $L^\infty(0,T;L^2(\Omega)) \hookrightarrow L^2(\Omega_T)$, since $L^1(0,T;L^{\frac{3}{4}}(\Omega))$ is the pre-dual of $L^\infty(0,T;L^4(\Omega))$, which is not reflexive Banach space, we have that

$$\varphi^k \rightharpoonup^* \varphi, \quad \text{in} \quad L^\infty(0,T;L^4(\Omega)).$$

Therefore, we get to φ is uniformly bounded in $L^\infty(0,T;L^4(\Omega))$.

Estimate IV. Setting $\eta^k = (\vartheta^k + \varphi^k)^3$ in (15), $\eta^k = (\vartheta^k + \varphi^k)^3$ in (16), and suppose that $\mu = \mu_1 = \mu_2 + \mu_3$, summing the resulting equations, and adding and subtracting the terms $(\varphi^k, (\vartheta^k + \varphi^k)^3)$, we deduce

$$\frac{1}{4}\frac{d}{dt}\|\vartheta^k + \varphi^k\|_{0,4}^4 + 3\mu\|(\vartheta^k + \varphi^k)\nabla(\vartheta^k + \varphi^k)\|_0^2 + \|\vartheta^k + \varphi^k\|_{0,4}^4 = (\varphi^k, (\vartheta^k + \varphi^k)^3). \tag{40}$$

We use (11) to find that

$$(\varphi^k, (\vartheta^k + \varphi^k)^3) \leq \frac{3}{4}\|\vartheta^k + \varphi^k\|_0^4 + \frac{1}{4}\|\varphi^k\|_{0,4}^4. \tag{41}$$

By substituting (41) in (40), and multiplying the result by 4, we conclude that

$$\frac{d}{dt}\|\vartheta^k + \varphi^k\|_{0,4}^4 + 12\mu\|(\vartheta^k + \varphi^k)\nabla(\vartheta^k + \varphi^k)\|_0^2 + \|\vartheta^k + \varphi^k\|_{0,4}^4 \leq \|\varphi^k\|_{0,4}^4. \tag{42}$$

Integral over time leads to

$$\begin{aligned}\|\vartheta^k + \varphi^k(T)\|_{0,4}^4 + 12\mu\|(\vartheta^k + \varphi^k)\nabla(\vartheta^k + \varphi^k)\|_{L^2(\Omega_T)}^2 + \|\vartheta^k + \varphi^k\|_{L^4(\Omega_T)}^4 \\ \leq \|\varphi^k\|_{L^4(\Omega_T)}^4 + \|\vartheta^k + \varphi^k(0)\|_{0,4}^4.\end{aligned} \tag{43}$$

Recalling $\vartheta_0^k, \varphi_0^k \in H^1(\Omega)$, using **Estimate III**, $L^\infty(0,T;L^4(\Omega)) \hookrightarrow L^4(\Omega_T)$ and since $L^1(0,T;L^{\frac{3}{4}}(\Omega))$ is the pre-dual of $L^\infty(0,T;L^4(\Omega))$, which is not reflexive Banach space, we have that

$$\vartheta^k + \varphi^k \rightharpoonup^* \vartheta + \varphi, \quad \text{in} \quad L^\infty(0,T;L^4(\Omega)).$$

Therefore, we get $\vartheta + \varphi$ is uniformly bounded in $L^\infty(0,T;L^4(\Omega))$, and since $L^4(\Omega_T)$ is a reflexive Banach space [19], then, by compactness arguments [6], we get the existence of subsequences $\vartheta^k + \varphi^k \in L^4(\Omega_T)$ such that

$$\vartheta^k + \varphi^k \rightharpoonup \vartheta + \varphi, \quad \text{in} \quad L^4(\Omega_T).$$

Estimate V. On setting $\eta^k = (\vartheta^k)^3$ in (15) and adding and subtracting $(\vartheta^k, (\vartheta^k)^5)$, we obtain that

$$\frac{1}{4}\frac{d}{dt}\|\vartheta^k\|_{0,4}^4 + 3\mu_1\|\vartheta^k\nabla\vartheta^k\|_0^2 + (\alpha+1)\|\vartheta^k\|_{0,4}^4 + \|\vartheta^k\|_{0,6}^6$$
$$= -3\mu_2(\nabla\varphi^k, (\vartheta^k)^2\nabla\vartheta^k) + (\vartheta^k + \varphi^k, (\vartheta^k)^5). \quad (44)$$

Using (7) and (11) gives that

$$\begin{aligned}
-3\mu_2(\nabla\varphi^k, (\vartheta^k)^2\nabla\vartheta^k) =& -3\mu_2\int_0^T \nabla\varphi^k\nabla\vartheta^k(\vartheta^k)^2 dx\\
\leq& 3\mu_2\|\vartheta^k\|_{0,\infty}^2\|\nabla\varphi^k\|_0\|\nabla\vartheta^k\|_0\\
\leq& \frac{9\mu_2^2}{2}\|\vartheta^k\|_{0,\infty}^4\|\nabla\varphi^k\|_0^2 + \frac{1}{2}\|\nabla\vartheta^k\|_0^2.
\end{aligned} \quad (45)$$

By using (7) Gagliardo - Nirenberg (9), and (11) on (44), we find that

$$\begin{aligned}
(\vartheta^k + \varphi^k, (\vartheta^k)^5) \leq& \|\vartheta^k + \varphi^k\|_{0,6}\|\vartheta^k\|_{0,6}^5\\
\leq& A\|\vartheta^k + \varphi^k\|_{0,4}^{\frac{2}{3}}\|\vartheta^k + \varphi^k\|_1^{\frac{1}{3}}\|\vartheta^k\|_{0,6}^5\\
\leq& A\|\vartheta^k + \varphi^k\|_{0,4}^4\|\vartheta^k + \varphi^k\|_1^2 + \frac{3}{4}\|\vartheta^k\|_{0,6}^6.
\end{aligned} \quad (46)$$

By substituting (46) and (45) into (44) and multiplying the result by 4, we have that

$$\frac{d}{dt}\|\vartheta^k\|_{0,4}^4 + 12\mu_1\|\vartheta^k\nabla\vartheta^k\|_0^2 + 4(\alpha+1)\|\vartheta^k\|_{0,4}^4 + \|\vartheta^k\|_{0,6}^6$$
$$\leq 18\mu_2^2\|\vartheta^k\|_{0,\infty}^4\|\nabla\varphi^k\|_0^2 + \frac{1}{2}\|\nabla\vartheta^k\|_0^2 + A\|\vartheta^k + \varphi^k\|_{0,4}^4\|\vartheta^k + \varphi^k\|_1^2. \quad (47)$$

Integrating over time leads to

$$\begin{aligned}
&\|\vartheta^k(T)\|_{0,4}^4 + 12\mu_1\|\vartheta^k\nabla\vartheta^k\|_{L^2(\Omega_T)}^2 + 4(\alpha+1)\|\vartheta^k\|_{L^4(\Omega_T)}^4 + \|\vartheta^k\|_{L^6(\Omega_T)}^6\\
&\leq 18\mu_2^2\max_{0\leq t\leq T}\|\vartheta^k\|_{0,\infty}^4\|\nabla\varphi^k\|_{L^2(\Omega_T)}^2 + 2\|\vartheta^k\|_{L^2(0,T;H^1(\Omega))} + A\max_{0\leq t\leq T}\|\vartheta^k + \varphi^k\|_{0,4}^4\|\vartheta^k\\
&+ \varphi^k\|_{L^2(0,T;H^1(\Omega))}^2 + \|\vartheta^k(0)\|_{0,4}^4.
\end{aligned} \quad (48)$$

Recalling $\vartheta_0^k \in H^1(\Omega)$, using **Estimate I**, $L^\infty(0,T;H^1(\Omega)) \hookrightarrow L^\infty(\Omega_T)$, **Estimate IV** and (17), since $L^1(0,T;L^{\frac{3}{4}}(\Omega))$ is the pre-dual of $L^\infty(0,T;L^4(\Omega))$, which is not reflexive Banach space, we have that

$$\vartheta^k \rightharpoonup^* \vartheta, \quad \text{in} \quad L^\infty(0,T;L^4(\Omega)).$$

Therefore, we get ϑ is uniformly bounded in $L^\infty(0,T;L^4(\Omega))$. We conclude that ϑ^k are uniformly bounded in $L^6(\Omega_T)$. Since $L^6(\Omega_T)$ is a reflexive Banach space [19], then, by compactness arguments [6], we get the existence of subsequences $\vartheta^k \in L^6(\Omega_T)$ such that

$$\vartheta^k \rightharpoonup \vartheta, \quad \text{in} \quad L^6(\Omega_T).$$

Estimate VI. By setting $\eta^k = (\varphi^k)^5$ in (16), it follows that

$$\frac{1}{6}\frac{d}{dt}\|\varphi^k\|_{0,6}^6 + 5\mu_3\|(\varphi^k)^2\nabla\varphi^k\|_0^2 + \|\vartheta^k(\varphi^k)^3\|_0^2 = \alpha(\vartheta^k, (\varphi^k)^5). \quad (49)$$

By using (11) into (49), we have that

$$\alpha(\vartheta^k,(\varphi^k)^5) \le \frac{1}{2}\|\vartheta^k(\varphi^k)^3\|_0^2 + \frac{\alpha^2}{2}\|\varphi^k\|_0^4. \quad (50)$$

By substituting (50) into (49), multiplying the result by 6, and integral over time, we obtain that

$$\|\varphi^k(T)\|_{0,6}^6 + 30\mu_3\|(\varphi^k)^2\nabla\varphi^k\|_{L^2(\Omega_T)}^2 + 3\|\vartheta^k(\varphi^k)^3\|_{L^2(\Omega_T)}^2 \le 3\alpha^2\|\varphi^k\|_{L^4(\Omega_T)}^4 + \|\varphi^k(0)\|_{0,6}^6. \quad (51)$$

Recalling $\varphi_0^k \in H^1(\Omega)$, using **Estimate III**, and $L^\infty(0,T;L^4(\Omega)) \hookrightarrow L^4(\Omega_T)$, since $L^1(0,T;L^{\frac{5}{6}}(\Omega))$ is the pre-dual of $L^\infty(0,T;L^6(\Omega))$, which is not reflexive Banach space, we have that

$$\vartheta^k \rightharpoonup^* \vartheta, \quad \text{in} \quad L^\infty(0,T;L^6(\Omega)).$$

Therefore, we get ϑ is uniformly bounded in $L^\infty(0,T;L^6(\Omega))$.

Estimate VII. Setting $\eta^k = (\vartheta^k+\varphi^k)^5$ in (15), $\eta^k = (\vartheta^k+\varphi^k)^5$ in (16) and suppose that $\mu = \mu_1 = \mu_2+\mu_3$, summing the resulting equations, and adding and subtracting the terms $(\varphi^k,(\vartheta^k+\varphi^k)^5)$, we deduce

$$\frac{1}{6}\frac{d}{dt}\|\vartheta^k+\varphi^k\|_{0,6}^6 + 5\mu\|(\vartheta^k+\varphi^k)^2\nabla(\vartheta^k+\varphi^k)\|_0^2 + \|\vartheta^k+\varphi^k\|_{0,6}^6 = (\varphi^k,(\vartheta^k+\varphi^k)^5). \quad (52)$$

We use (11) to have

$$(\varphi^k,(\vartheta^k+\varphi^k)^5) \le \frac{5}{6}\|\vartheta^k+\varphi^k\|_{0,6}^6 + \frac{1}{6}\|\varphi^k\|_{0,6}^6. \quad (53)$$

By substituting (53) in (52), and multiplying the result by 6, it follows that

$$\frac{d}{dt}\|\vartheta^k+\varphi^k\|_{0,6}^6 + 30\mu\|(\vartheta^k+\varphi^k)^2\nabla(\vartheta^k+\varphi^k)\|_0^2 + \|\vartheta^k+\varphi^k\|_{0,6}^6 \le \|\varphi^k\|_{0,6}^4. \quad (54)$$

Integrating over time leads to

$$\|\vartheta^k+\varphi^k(T)\|_{0,6}^6 + 30\mu\|(\vartheta^k+\varphi^k)^2\nabla(\vartheta^k+\varphi^k)\|_{L^2(\Omega_T)}^2 + \|\vartheta^k+\varphi^k\|_{L^6(\Omega_T)}^6 \\ \le \|\varphi^k\|_{L^6(\Omega_T)}^6 + \|\vartheta^k+\varphi^k(0)\|_{0,6}^6. \quad (55)$$

Recalling $\vartheta_0^k, \varphi_0^k \in H^1(\Omega)$, since $L^1(0,T;L^{\frac{5}{6}}(\Omega))$ is the pre-dual of $L^\infty(0,T;L^6(\Omega))$, which is not reflexive Banach space, we have that

$$\vartheta^k+\varphi^k \rightharpoonup^* \vartheta+\varphi, \quad \text{in} \quad L^\infty(0,T;L^6(\Omega)).$$

Therefore, we get $\vartheta+\varphi$ is uniformly bounded in $L^\infty(0,T;L^6(\Omega))$, and since $L^6(\Omega_T)$ is a reflexive Banach space [19], then, by compactness arguments [6], we get the existence of subsequences $\vartheta^k+\varphi^k \in L^6(\Omega_T)$ such that

$$\vartheta^k+\varphi^k \rightharpoonup \vartheta+\varphi, \quad \text{in} \quad L^6(\Omega_T).$$

Estimate VIII. On setting $\eta^k = (\vartheta^k)^5$ in (15) and adding $(\vartheta^k,(\vartheta^k)^7)$ for both sides, we obtain that

$$\frac{1}{6}\frac{d}{dt}\|\vartheta^k\|_{0,6}^6 + 5\mu_1\|(\vartheta^k)^2\nabla\vartheta^k\|_0^2 + (\alpha+1)\|\vartheta^k\|_{0,6}^6 + \|\vartheta^k\|_{0,8}^8 \\ = -5\mu_2(\nabla\varphi^k,(\vartheta^k)^4\nabla\vartheta^k) + (\vartheta^k+\varphi^k,(\vartheta^k)^7). \quad (56)$$

Using (7) and (11), we find that

$$\begin{aligned}-5\mu_2(\nabla\varphi^k,(\vartheta^k)^4\nabla\vartheta^k) =& -5\mu_2\int_0^T \nabla\varphi^k\nabla\vartheta^k(\vartheta^k)^4dx\\ \leq& 5\mu_2\|\vartheta^k\|_{0,\infty}^4\|(\nabla\varphi^k\|_0\|\nabla\vartheta^k\|_0\\ \leq& \frac{25\mu_2^2}{2}\|\vartheta^k\|_{0,\infty}^4\|\nabla\varphi^k\|_0^2+\frac{1}{2}\|\nabla\vartheta^k\|_0^2.\end{aligned} \tag{57}$$

By using (7), the Gagliardo-Nirenberg (9) and (11) on the right hand side of (56), we find that

$$\begin{aligned}(\vartheta^k+\varphi^k,(\vartheta^k)^7) \leq& \|\vartheta^k+\varphi^k\|_{0,8}\|\vartheta^k\|_{0,8}^7\\ \leq& A\|\vartheta^k+\varphi^k\|_{0,6}^{\frac{3}{4}}\|\vartheta^k+\varphi^k\|_1^{\frac{1}{4}}\|\vartheta^k\|_{0,8}^6\\ \leq& A\|\vartheta^k+\varphi^k\|_{0,6}^6\|\vartheta^k+\varphi^k\|_1^2+\frac{5}{6}\|\vartheta^k\|_{0,8}^8.\end{aligned} \tag{58}$$

By substituting (57) and (58) into (56) and multiplying the result by 6, we obtain

$$\begin{aligned}&\frac{d}{dt}\|\vartheta^k\|_{0,6}^6+30\mu_1\|(\vartheta^k)^2\nabla\vartheta^k\|_0^2+6(\alpha+1)\|\vartheta^k\|_{0,6}^6+\|\vartheta^k\|_{0,8}^8\\ &\qquad\leq 75\mu_2^2\|\vartheta^k\|_{0,\infty}^4\|\nabla\varphi^k\|_0^2+3\|\nabla\vartheta^k\|_0^2+A\|\vartheta^k+\varphi^k\|_{0,6}^6\|\vartheta^k+\varphi^k\|_1^2.\end{aligned} \tag{59}$$

Integrating over time leads to

$$\begin{aligned}&\|\vartheta^k(T)\|_{0,6}^6+30\mu_1\|(\vartheta^k)^2\nabla\vartheta^k\|_{L^2(\Omega_T)}^2+6(\alpha+1)\|\vartheta^k\|_{L^6(\Omega_T)}^6+\|\vartheta^k\|_{L^8(\Omega_T)}^8\\ &\leq 75\mu_2^2\max_{0\leq t\leq T}\|\vartheta^k\|_{0,\infty}^4\|\nabla\varphi^k\|_{L^2(\Omega_T)}^2+3\|\vartheta^k\|_{L^2(0,T;H^1(\Omega))}+A\max_{0\leq t\leq T}\|\vartheta^k+\varphi^k\|_{0,6}^6\|\vartheta^k\\ &+\varphi^k\|_{L^2(0,T;H^1(\Omega))}^2+\|\vartheta^k(0)\|_{0,6}^6.\end{aligned} \tag{60}$$

Recalling $\vartheta_0^k\in H^1(\Omega)$, using **Estimate I**, $L^\infty(0,T;H^1(\Omega))\hookrightarrow L^\infty(\Omega_T)$, **Estimate VII** and (17), since $L^1(0,T;L^{\frac{5}{6}}(\Omega))$ is the pre-dual of $L^\infty(0,T;L^6(\Omega))$, which is not reflexive Banach space, we have that

$$\vartheta^k\rightharpoonup^*\vartheta,\quad \text{in}\quad L^\infty(0,T;L^6(\Omega)).$$

Therefore, we get ϑ is uniformly bounded in $L^\infty(0,T;L^6(\Omega))$, and since $L^8(\Omega_T)$ is a reflexive Banach space [19], then, by compactness arguments [6], we get the existence of subsequences $\vartheta^k\in L^8(\Omega_T)$ such that

$$\vartheta^k\rightharpoonup\vartheta,\quad \text{in}\quad L^8(\Omega_T).$$

Lemma 3.1. *If $k\geq 0$, assume that*

$$\vartheta\in L^2(0,T;H^{k+1}(\Omega)),\quad \frac{\partial\vartheta}{\partial t}\in L^2(0,T;H^{k-1}(\Omega)).$$

Then, $\vartheta\in C([0,T];H^1(\Omega))$.

Proof. See [13], pages 191-194. □

Here, in our case, $k=1$, $H^{k+1}(\Omega)=H^2(\Omega)$, $H^k(\Omega)=H^1(\Omega)$, $H^{k-1}(\Omega)=L^2(\Omega)$. Thus, from Lemma 3.1 we have that $\vartheta,\varphi\in C([0,T];H^1(\Omega))$.

3.2 Continuous dependence

Suppose that $\{\vartheta_1, \varphi_1\}$ and $\{\vartheta_2, \varphi_2\}$ satisfy the weak form (15) and (16), with initial conditions $\vartheta_1(.,0) = \vartheta_{1,0}(.), \vartheta_2(.,0) = \vartheta_{2,0}(.)$, and $\varphi_1(.,0) = \varphi_{1,0}(.), \varphi_2(.,0) = \varphi_{2,0}(.)$, respectively such that $\vartheta_{1,0}(.) \neq \vartheta_{2,0}(.)$, and $\varphi_{1,0}(.) \neq \varphi_{2,0}(.)$. Setting $\varrho_1 = \vartheta_1 - \vartheta_2$ and $\varrho_2 = \varphi_1 - \varphi_2$, and setting $\varpi = -\Delta\varrho_1 + \varrho_1$ and $\varpi = -\Delta\varrho_2 + \varrho_2$ in (15) and (16), subtracting weak forms lead after integrating by parts to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\varrho_1\|_0^2 + \|\nabla\varrho_1\|_0^2\Big) + \frac{1}{2}\frac{d}{dt}\Big(\|\varrho_2\|_0^2 + \|\nabla\varrho_2\|_0^2\Big) + \mu_1\big(\|\nabla\varrho_1\|_0^2 + \|\Delta\varrho_1\|_0^2\big)\\
&+ \mu_3\big(\|\nabla\varrho_2\|_0^2 + \|\Delta\varrho_2\|_0^2\big) + (\alpha+1)\big(\|\nabla\varrho_1\|_0^2 + \|\varrho_1\|_0^2\big)\\
&= \alpha(\nabla\varrho_1\nabla\varrho_2 + \varrho_1\varrho_2) + \mu_2\int_\Omega(\Delta\varrho_2\Delta\varrho_1 + \nabla\varrho_2\nabla\varrho_1)dx\\
&+ (\varphi_1\vartheta_1^2 - \varphi_2\vartheta_2^2, -\Delta\varrho_1 + \varrho_1) - (\varphi_1\vartheta_1^2 - \varphi_2\vartheta_2^2, -\Delta\varrho_2 + \varrho_2).
\end{aligned} \tag{61}$$

By applying (11) into (61), and obtain

$$\alpha(\nabla\varrho_1\nabla\varrho_2 + \varrho_1\varrho_2) \leq \frac{\alpha^2}{\mu_1}\|\nabla\varrho_2\|_0^2 + \frac{\mu_1}{4}\|\nabla\varrho_1\|_0^2 + \frac{\alpha+1}{2}\|\varrho_1\|_0^2 + \frac{\alpha^2}{2(\alpha+1)}\|\varrho_2\|_0^2. \tag{62}$$

By applying (7) and (11), yield

$$\begin{aligned}
&(\varphi_1\vartheta_1^2 - \varphi_2\vartheta_2^2, \varrho_1) - (\varphi_1\vartheta_1^2 - \varphi_2\vartheta_2^2, \varrho_2)\\
&= (\varphi_1(\vartheta_1^2 - \vartheta_2^2) + (\varphi_1 - \varphi_2)\vartheta_2^2, \varrho_1) - (\varphi_1(\vartheta_1^2 - \vartheta_2^2) + (\varphi_1 - \varphi_2)\vartheta_2^2, \varrho_2)\\
&= (\varphi_1(\vartheta_1 + \vartheta_2)\varrho_1, \varrho_1) + (\varrho_2\vartheta_2^2, \varrho_1) - (\varphi_1(\vartheta_1 + \vartheta_2)\varrho_1, \varrho_2) - (\varrho_2\vartheta_2^2, \varrho_2)\\
&\leq C\big[\|\varphi_1\|_{0,\infty}^2 + \|\vartheta_1\|_{0,\infty}^2 + \|\vartheta_2\|_{0,\infty}^2\big]\big[\|\varrho_1\|_0^2 + \|\varrho_2\|_0^2\big] - \|\vartheta_2\varrho_2\|_0^2.
\end{aligned} \tag{63}$$

In the same way, we can show that

$$\begin{aligned}
&-(\varphi_1\vartheta_1^2 - \varphi_2\vartheta_2^2, \Delta\varrho_1) + (\varphi_1\vartheta_1^2 - \varphi_2\vartheta_2^2, \Delta\varrho_2)\\
&= (\varphi_1\varrho_2(\vartheta_1 + \vartheta_2), \Delta\varrho_1) - (\vartheta_2^2\varrho_2, \Delta\varrho_1) - (\varphi_1\varrho_1(\vartheta_1 + \vartheta_2), \Delta\varrho_2) + (\vartheta_2^2\varrho_2, \Delta\varrho_2)\\
&\leq C\big[\|\varphi_1\|_{0,\infty}^2 + \|\vartheta_1\|_{0,\infty}^2 + \|\vartheta_2\|_{0,\infty}^2\big]\big[\|\nabla\varrho_1\|_0^2 + \|\nabla\varrho_2\|_0^2\big] - \|\vartheta_2\|_{0,\infty}^2\|\nabla\varrho_2\|_0^2.
\end{aligned} \tag{64}$$

Substituting (62)-(64) into (61) leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\varrho_1\|_0^2 + \|\nabla\varrho_1\|_0^2\Big) + \frac{1}{2}\frac{d}{dt}\Big(\|\varrho_2\|_0^2 + \|\nabla\varrho_2\|_0^2\Big) + (\mu_1 - \frac{\mu_2^2}{2\mu_3})\|\Delta\varrho_1\|_0^2 + \frac{1}{2}\mu_1\|\nabla\varrho_1\|_0^2\\
&+ \frac{1}{2}\mu_3\|\Delta\varrho_2\|_0^2 + (\mu_3 - \frac{\alpha^2}{\mu_1} - \frac{\mu_2^2}{\mu_1})\|\nabla\varrho_2\|_0^2 + (\alpha+1)\|\nabla\varrho_1\|_0^2 + \frac{\alpha+1}{2}\|\varrho_1\|_0^2 + \|\vartheta_2\varrho_2\|_0^2\\
&+ \|\vartheta_2\|_{0,\infty}^2\|\nabla\varrho_2\|_0^2\\
&\leq \frac{\alpha^2}{2(\alpha+1)}\|\varrho_2\|_0^2 + C\big[\|\varphi_1\|_{0,\infty}^2 + \|\vartheta_1\|_{0,\infty}^2 + \|\vartheta_2\|_{0,\infty}^2\big]\big[\|\varrho_1\|_0^2 + \|\nabla\varrho_1\|_0^2 + \|\varrho_2\|_0^2 + \|\nabla\varrho_2\|_0^2\big].
\end{aligned} \tag{65}$$

If we eliminate the positive terms from the left-hand side and multiply the result by 2, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\varrho_1\|_0^2 + \frac{d}{dt}\|\varrho_2\|_0^2\\
&\quad\leq C\big[1 + \|\varphi_1\|_{0,\infty}^2 + \|\vartheta_1\|_{0,\infty}^2 + \|\vartheta_2\|_{0,\infty}^2\big] + \big[\|\varrho_1\|_{0,\infty}^2 + \|\varrho_2\|_{0,\infty}^2 + \|\nabla\varrho_1\|_0^2 + \|\nabla\varrho_2\|_0^2\big].
\end{aligned} \tag{66}$$

From the application of the Grönwall lemma (10), yields

$$\|\varrho_1(T)\|_1^2 + \|\varrho_2(T)\|_1^2 \\ \leq \exp\left(2CT + \int_0^T \left[1 + \|\varphi_1\|_{0,\infty}^2 + \|\vartheta_1\|_{0,\infty}^2 + \|\vartheta_2\|_{0,\infty}^2\right] dt\right) \left[\|\varrho_1(0)\|_1^2 + \|\varrho_2(0)\|_1^2\right]. \quad (67)$$

On noting the uniform bounds in (17), we have

$$\|\varrho_1(T)\|_1^2 + \|\varrho_2(T)\|_1^2 \leq C\big(\|\varrho_1(0)\|_1^2 + \|\varrho_2(0)\|_1^2\big). \quad (68)$$

Thus, if $(\vartheta_1(0), \varphi_1(0)) = (\vartheta_2(0), \varphi_2(0))$, then $(\varrho_2(0), \varrho_2(0)) = (0, 0)$ and hence it follows from (68) that $(\varrho_2(t), \varrho_2(t)) = (0, 0)$ and hence $\vartheta_1(t) = \vartheta_2(t)$ and $\varphi_1(t) = \varphi_2(t)$ for all t. However, if $(\vartheta_1(0), \varphi_1(0)) \neq (\vartheta_2(0), \varphi_2(0))$, then we have continuous dependence in $H^1(\Omega)$. This is complete proof.

4 Conclusions

If the initial data is in $H^1(\Omega)$, there is a unique global strong solution depending continuously on the initial data. We show the continuous dependence of the strong solution on the initial data. This seems to represent a limitation in the Faedo-Galerkin method and the fact when we took the initial data in $H^1(\Omega)$.

Acknowledgement We express our gratitude to Editor-in-Chief and anonymous referees for their insightful comments, which have significantly contributed to enhancing the quality of the manuscript.

Conflicts of Interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] R. A. Adams & J. J. F. Fournier (2003). *Sobolev spaces*. Elsevier, Amsterdam.

[2] G. A. Al-Juaifri & A. J. Harfash (2023). Analysis of a nonlinear reaction-diffusion system of the Fitzhugh-Nagumo type with Robin boundary conditions. *Ricerche di Matematica*, *72*(1), 335–357. https://doi.org/10.1007/s11587-022-00711-7.

[3] G. A. Al-Juaifri & A. J. Harfash (2023). Finite element analysis of nonlinear reaction-diffusion system of Fitzhugh-Nagumo type with Robin boundary conditions. *Mathematics and Computers in Simulation*, *203*, 486–517. https://doi.org/10.1016/j.matcom.2022.07.005.

[4] G. A. Al-Juaifri & A. J. Harfash (2024). Existence and uniqueness of solution for the nonlinear Brusselator system with Robin boundary conditions. *Georgian Mathematical Journal*, *31*(3), 355–368. https://doi.org/10.1515/gmj-2023-2091.

[5] P. G. Ciarlet (2002). *The finite element method for elliptic problems*. Society for Industrial and Applied Mathematics, Philadelphia, USA. https://doi.org/10.1137/1.9780898719208.

[6] R. Dautray & J.-L. Lions (1999). *Mathematical analysis and numerical methods for science and technology* volume 2. Springer, New York, USA.

[7] P. Grisvard (2011). *Elliptic problems in nonsmooth domains*. Society for Industrial and Applied Mathematics, Philadelphia, USA. https://doi.org/10.1137/1.9781611972030.

[8] W. Han (2013). Life-span of classical solutions to one dimensional initial-boundary value problems for general quasilinear wave equations with Robin boundary conditions. *Journal of Mathematical Analysis and Applications*, *397*(1), 46–74. https://doi.org/10.1016/j.jmaa.2011.08.071.

[9] S. Haq, I. Ali & K. S. Nisar (2021). A computational study of two-dimensional reaction-diffusion Brusselator system with applications in chemical processes. *Alexandria Engineering Journal*, *60*(5), 4381–4392. https://doi.org/10.1016/j.aej.2021.02.064.

[10] A. Kolinichenko & L. Ryashko (2020). Multistability and stochastic phenomena in the distributed Brusselator model. *Journal of Computational and Nonlinear Dynamics*, *15*(1), 011007. https://doi.org/10.1115/1.4045405.

[11] N. M. Noor, S. A. M. Yatim & Z. B. Ibrahim (2024). Fractional block method for the solution of fractional order differential equations. *Malaysian Journal of Mathematical Sciences*, *18*(1), 185–208. https://doi.org/10.47836/mjms.18.1.11.

[12] I. Prigogine & R. Lefever (1968). Symmetry breaking instabilities in dissipative systems. II. *The Journal of Chemical Physics*, *48*(4), 1695–1700. https://doi.org/10.1063/1.1668896.

[13] J. C. Robinson (2001). *Infinite-dimensional dynamical systems: An introduction to dissipative parabolic PDEs and the theory of global attractors* volume 28. Cambridge University Press, Cambridge, United Kingdom.

[14] M. M. Salih & F. Ismail (2022). Trigonometrically-fitted fifth order four-step predictor-corrector method for solving linear ordinary differential equations with oscillatory solutions. *Malaysian Journal of Mathematical Sciences*, *16*(4), 739–748. https://doi.org/10.47836/mjms.16.4.07.

[15] J. A. Sherratt (2008). A comparison of periodic travelling wave generation by Robin and Dirichlet boundary conditions in oscillatory reaction-diffusion equations. *IMA Journal of Applied Mathematics*, *73*(5), 759–781. https://doi.org/10.1093/imamat/hxn015.

[16] R. Temam (2012). *Infinite-dimensional dynamical systems in mechanics and physics* volume 68. Springer Science & Business Media, New York. https://doi.org/10.1007/978-1-4612-0645-3.

[17] A. M. Turing (1952). The chemical basis of morphogenesis. *Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences*, *237*(641), 37–72. https://doi.org/10.1098/rstb.1952.0012.

[18] J. J. Tyson (1973). Some further studies of nonlinear oscillations in chemical systems. *The Journal of Chemical Physics*, *58*(9), 3919–3930. https://doi.org/10.1063/1.1679748.

[19] A. Ženíšek & J. R. Whiteman (1990). *Nonlinear elliptic and evolution problems and their finite element aproximations (computational mathematics and applications)*. Academic Press, London.